

Continuous Solutions Creation and Destruction Operators Derivation

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1 Delta Function Units

Note first that the Dirac delta function has inverse units of its argument. For example, in

$$\int \delta(x - x') dx = 1 \text{ (unitless)} \quad \int f(x) \delta(x - x') dx = f(x') \text{ (units of function } f) , \quad (1)$$

$\delta(x - x')$ has units of $1/x$ (cm^{-1} in cgs units, M^{-1} in natural units). Similarly, $\delta(k - k')$ has units of $1/k$ (cm in cgs, since units of wave number k are the inverse of length units; M^{-1} in natural units).

In 3D, $\delta^{(3)}(\mathbf{x} - \mathbf{x}')$ [often just written as $\delta(\mathbf{x} - \mathbf{x}')$] has units of $1/\text{volume}$ (cm^{-3} in cgs, M^{-3} in natural units); $\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ [often just written as $\delta(\mathbf{k} - \mathbf{k}')$] has units of volume (cm^3 in cgs, M^{-3} in natural units).

2 Creation and Destruction Operator Units

Thus, in the 3D relation for the coefficients of the continuous solutions,

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}') , \quad (2)$$

the delta function has units of $1/\mathbf{k}^3$ (M^{-3} in natural units), so the $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ operators each have units of $1/\sqrt{\mathbf{k}^3}$ ($M^{-3/2}$ in natural units).

So, in

$$\begin{aligned} \phi(x) &= \underbrace{\int \frac{d^3\mathbf{k}}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx}}_{\phi^+} + \underbrace{\int \frac{d^3\mathbf{k}}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} b^\dagger(\mathbf{k}) e^{ikx}}_{\phi^-} \quad (a) \\ &= \phi^+ + \phi^- \quad (3-37) \text{ in text} \quad (3) \end{aligned}$$

$$\begin{aligned} \phi^\dagger(x) &= \underbrace{\int \frac{d^3\mathbf{k}}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} b(\mathbf{k}) e^{-ikx}}_{\phi^{\dagger+}} + \underbrace{\int \frac{d^3\mathbf{k}}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} a^\dagger(\mathbf{k}) e^{ikx}}_{\phi^{\dagger-}} \quad (b) \\ &= \phi^{\dagger+} + \phi^{\dagger-} , \end{aligned}$$

given that $\omega_{\mathbf{k}}$ and \mathbf{k} have the same units, and $a(\mathbf{k})$ has units as shown just above (3), then ϕ has units of \mathbf{k} (M^1 in natural units). $b(\mathbf{k})$

Note that ϕ having units of \mathbf{k} (M^1) in the discrete case

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V \omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx} + \sum_{\mathbf{k}} \frac{1}{\sqrt{2V \omega_{\mathbf{k}}}} b^\dagger(\mathbf{k}) e^{ikx} , \quad (3-37a) \text{ in text,} \quad (4)$$

means that $a(\mathbf{k})$ in that case is unitless. This follows from the commutation relation

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}') \right] = \delta_{\mathbf{k}\mathbf{k}'} \quad \text{discrete case,} \quad (5)$$

since $\delta_{\mathbf{k}\mathbf{k}'}$ is unitless.

3 Hamiltonian for Continuous Solutions

From www.quantumfieldtheory.info/H_operator_and_noneigen_states.pdf, equation (37) therein, for continuous solutions, the Hamiltonian operator is

$$H = \int \omega_{\mathbf{k}} \left(\mathcal{N}_a(\mathbf{k}) + \frac{1}{2} \delta(0) + \mathcal{N}_b(\mathbf{k}) + \frac{1}{2} \delta(0) \right) d^3k \quad (10-8) \text{ in } SFQFT \text{ text.} \quad (6)$$

$$\mathcal{N}_a(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) \quad \mathcal{N}_b(\mathbf{k}) = b^\dagger(\mathbf{k})b(\mathbf{k}) \quad (\text{continuous})$$

For the vacuum terms in (6), the $\delta(0)$ factor of infinity has units of $1/(\mathbf{k} \text{ space vol})$, or equivalently, since $|\mathbf{k}| = 2\pi/\lambda$ (λ is wavelength), units of \mathbf{x} space volume. The $\delta(0)$ represents the infinite volume of space over which our Hamiltonian H represents the total energy. Thus, the \mathbf{x} (physical) space Hamiltonian volume density for the vacuum energy is

$$\mathcal{H}_{vac} = \frac{H_{vac}}{V_\infty} = \int \omega_{\mathbf{k}} \left(\frac{1}{2} + \frac{1}{2} \right) d^3k = \int \omega_{\mathbf{k}} d^3k. \quad (7)$$

Note, from (6), that the vacuum terms factor of $1/2$ is dimensionless (no units). That is,

$$\frac{1}{2} = \left((\text{number of vacuum particle states}) / (\text{unit vol in } \mathbf{k} \text{ space}) \right) / (\text{unit vol in } \mathbf{x} \text{ space}), \quad (8)$$

$$\text{dimensions } M^3 / M^3 = \text{dimensionless}.$$

4 Number Operator Units

For the number operator relations

$$N_a(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) \quad (\text{discrete}) \quad \mathcal{N}_a(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) \quad (\text{continuous}), \quad (9)$$

we know that $N_a(\mathbf{k})$ is unitless (a number operator without dimensions). From the Hamiltonian operator for continuous solution scalars (6), we see that $\mathcal{N}_a(\mathbf{k})$ is a number density operator (density in \mathbf{k} space). That is,

$$\mathcal{N}_a(\mathbf{k}), \mathcal{N}_b(\mathbf{k}) = (\text{number of real particle states}) / \text{unit vol in } \mathbf{k} \text{ space, dimensions } 1/M^3 \quad (10)$$

These operators act on all states spread through all of \mathbf{x} space, as our integrations in \mathbf{x} space (as well as \mathbf{k} space) in the continuous solution approach are from $-\infty$ to $+\infty$ in all directions.

5 Creation and Destruction Operators Derived: Continuous Case

In parallel with the discrete solution derivation of Sect. 3.6.1, pg. 58, of *SFQFT*, consider the \mathbf{k} th wave component of a wave packet ket as $|n_{\mathbf{k}}\rangle$. Then

$$\mathcal{N}_a(\mathbf{k})|n_{\mathbf{k}}\rangle = n_{\mathbf{k}}|n_{\mathbf{k}}\rangle, \quad (11)$$

where $n_{\mathbf{k}}$ has the same units as $\mathcal{N}_a(\mathbf{k})$, i.e., (number of real particle states)/(\mathbf{k} space vol). Note carefully, that $n_{\mathbf{k}}$ is for all space (infinite spatial volume).

We then ask, what is

$$a(\mathbf{k})|n_{\mathbf{k}}\rangle = |m_{\mathbf{k}}\rangle ? \quad \text{parallel to (3-71) for discrete case in text} \quad (12)$$

$$\mathcal{N}_a(\mathbf{k})|m_{\mathbf{k}}\rangle = \mathcal{N}_a(\mathbf{k})a(\mathbf{k})|n_{\mathbf{k}}\rangle = \underbrace{a^\dagger(\mathbf{k})a(\mathbf{k})}_{\text{use commutator}}a(\mathbf{k})|n_{\mathbf{k}}\rangle \quad \text{parallel to (3-72)} \quad (13)$$

$$\begin{aligned} &= (a(\mathbf{k})a^\dagger(\mathbf{k}) - \delta(0))a(\mathbf{k})|n_{\mathbf{k}}\rangle = a(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k})|n_{\mathbf{k}}\rangle - \delta(0)a(\mathbf{k})|n_{\mathbf{k}}\rangle = a(\mathbf{k})N_a(\mathbf{k})|n_{\mathbf{k}}\rangle - \delta(0)a(\mathbf{k})|n_{\mathbf{k}}\rangle \\ &= a(\mathbf{k})n_{\mathbf{k}}|n_{\mathbf{k}}\rangle - \delta(0)a(\mathbf{k})|n_{\mathbf{k}}\rangle = n_{\mathbf{k}}a(\mathbf{k})|n_{\mathbf{k}}\rangle - \delta(0)a(\mathbf{k})|n_{\mathbf{k}}\rangle = (n_{\mathbf{k}} - \delta(0))a(\mathbf{k})|n_{\mathbf{k}}\rangle = (n_{\mathbf{k}} - \delta(0))|m_{\mathbf{k}}\rangle. \end{aligned} \quad (14)$$

So,

$$\mathcal{N}_a(\mathbf{k})|m_{\mathbf{k}}\rangle = (n_{\mathbf{k}} - \delta(0))|m_{\mathbf{k}}\rangle = m_{\mathbf{k}}|m_{\mathbf{k}}\rangle \quad m_{\mathbf{k}} = n_{\mathbf{k}} - \underbrace{\delta(0)}_{=V_\infty} \quad \text{parallel to (3-74)}. \quad (15)$$

In (15), $m_{\mathbf{k}}$ and $n_{\mathbf{k}}$ are for all space (infinite in size in our continuous solution math where all integrals are over $-\infty$ to $+\infty$). Thus, for any finite number (however small) of \mathbf{k} states per unit volume of space, $m_{\mathbf{k}}$ and $n_{\mathbf{k}}$ must be infinite.

If we define new quantities ${}_u n_{\mathbf{k}}$ and ${}_u m_{\mathbf{k}}$ as spatial densities, i.e., as number per unit spatial volume, such that

$$n_{\mathbf{k}} = {}_u n_{\mathbf{k}} V_\infty \quad m_{\mathbf{k}} = {}_u m_{\mathbf{k}} V_\infty \quad {}_u n_{\mathbf{k}}, {}_u m_{\mathbf{k}} = \frac{\text{number of } \mathbf{k} \text{ fields per unit } \mathbf{k} \text{ volume}}{\text{unit volume of space}}, \quad (16)$$

then the RHS of (15) becomes

$$m_{\mathbf{k}} = n_{\mathbf{k}} - V_\infty = {}_u m_{\mathbf{k}} V_\infty = {}_u n_{\mathbf{k}} V_\infty - V_\infty. \quad (17)$$

Dividing (17) by V_∞ we have

$${}_u m_{\mathbf{k}} = {}_u n_{\mathbf{k}} - 1, \quad (18)$$

and thus, from (15),

$$\frac{\mathcal{N}_a(\mathbf{k})}{V_\infty}|m_{\mathbf{k}}\rangle = ({}_u n_{\mathbf{k}} - 1)|m_{\mathbf{k}}\rangle = {}_u m_{\mathbf{k}}|m_{\mathbf{k}}\rangle \quad (19)$$

(18) and (19) remind us of (3-74) in *SFQFT*. But, whereas (3-74) implied $a(\mathbf{k})$ in the discrete solutions case decreased the total number of \mathbf{k} states by one, here in the continuous solutions case, $a(\mathbf{k})$ decreases the \mathbf{k} states per unit \mathbf{k} volume per unit \mathbf{x} volume by one.

Similar analysis leads to $a^\dagger(\mathbf{k})$ in the continuous state increasing the \mathbf{k} states per unit \mathbf{k} volume per unit \mathbf{x} volume by one. Thus, the continuous solutions coefficients $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are respectively, destruction and creation operators. By parallel logic, $b(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ behave in the same way for b type states.